

LIMIT OF GREEN FUNCTIONS AND IDEALS, THE CASE OF FOUR POLES

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ABSTRACT. We study the limits of pluricomplex Green functions with four poles tending to the origin in a hyperconvex domain, and the (related) limits of the ideals of holomorphic functions vanishing on those points. Taking subsequences, we always assume that the directions defined by pairs of points stabilize as they tend to 0. We prove that in a generic case, the limit of the Green functions is always the same, while the limits of ideals are distinct (in contrast to the three point case). We also study some exceptional cases, where only the limits of ideals are determined. In order to do this, we establish a useful result linking the length of the upper or lower limits of a family of ideals, and its convergence.

1. INTRODUCTION

The definition of multipole pluricomplex Green functions with logarithmic singularities, in the wake of Lempert's seminal work [6], was motivated by the nonlinearity of the complex Monge-Ampère equation, and generalizations of the Schwarz Lemma, see e.g. Demailly [1], [12], Lelong [5].

Sometimes it is useful to study the limit case where poles tend to each other [10], an analogue of multiple zeroes for holomorphic functions, and this leads naturally to the more general notion of the Green function of an ideal of holomorphic functions:

Definition 1.1. [8] *Let Ω be a hyperconvex bounded domain in \mathbb{C}^n , $\mathcal{O}(\Omega)$ the space of holomorphic functions on this domain.*

Let \mathcal{I} be an ideal of $\mathcal{O}(\Omega)$, and ψ_j its generators. Then

$$G_{\mathcal{I}}^{\Omega}(z) := \sup \left\{ u(z) : u \in PSH_{-}(\Omega), u(z) \leq \max_j \log |\psi_j| + O(1) \right\}.$$

Key words and phrases. pluricomplex Green function, complex Monge-Ampère equation, ideals of holomorphic functions.

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Note that the condition is meaningful only near $a \in V(\mathcal{I}) := \{p \in \Omega : f(p) = 0, \forall f \in \mathcal{I}\}$. Since the domain is pseudoconvex, there are finitely many global generators $\psi_j \in \mathcal{O}(\Omega)$ such that for any $f \in \mathcal{I}$, there exists $h_j \in \mathcal{O}(\Omega)$ such that $f = \sum_j h_j \psi_j$, see e.g. [4, Theorem 7.2.9, p. 190].

In the special case when S is a finite set in Ω and $\mathcal{I} = \mathcal{I}(S)$, the ideal of all functions vanishing on the set S (which we sometimes call point-based ideal), this reduces to a pluricomplex Green function with logarithmic singularities; we write $G_{\mathcal{I}(S)} = G_S$.

We want to study the limit of G_{S_ε} when S_ε is a set of points tending to the origin, and relate this to the limit of the ideals $\mathcal{I}(S_\varepsilon)$. It is a consequence of [9] that if convergence of those Green functions takes place in the (relatively weak) sense of $L^1_{\text{loc}}(\overline{\Omega})$, then that convergence is actually uniform on compacta of $\overline{\Omega} \setminus \{0\}$, so it will be understood that all convergence results are in this sense.

The case of 3 poles in dimension $n = 2$ was worked out in [7, Theorem 1.12, (i)]; a remaining subcase of that study was finally settled in [2].

In the present paper, we explore the case of 4 points tending to the origin in \mathbb{C}^2 . Unlike in the three-point case, where the limit ideal was generically \mathfrak{M}_0^2 and the limits of the Green functions depended on the directions along which the points tended to 0, here we will see that, generically (in a sense to be made precise), $\lim G_{\mathcal{I}_\varepsilon} = G_{\lim \mathcal{I}_\varepsilon}$, and that this limit is the same, namely, $\lim G_{\mathcal{I}_\varepsilon} = 2 \max(\log |z_1|, \log |z_2|) + O(1)$ (Theorem 2.1), whereas the limit ideals very much depend on the directions of convergence to 0.

Some singular cases are studied in Theorems 2.2 and 2.3, although here we mostly compute limits of ideals, the Green functions of which cannot coincide with the limit of our Green functions because of Theorem 4.2 below. The results of [9] are used to yield some estimates of the Green functions in those cases, but the complete answer is not known.

In order to obtain those results, we establish Theorem 2.5, an auxiliary result about convergence of ideals which shortens the proofs, and should be of independent interest.

2. STATEMENT OF THE RESULTS

2.1. Notations. As usual, $\mathfrak{M}_0 := \mathcal{I}(\{(0, 0)\})$ stands for the maximal ideal at $(0, 0)$, and $\mathfrak{M}_0^2, \mathfrak{M}_0^3 \dots$ for its successive powers. For an ideal $\mathcal{I} \subset \mathcal{O}(\Omega)$, its *length* (or co-length) is $\ell(\mathcal{I}) := \dim \mathcal{O}(\Omega) / \dim(\mathcal{I})$. For instance, $\ell(\mathfrak{M}_0^k) = \frac{1}{2}k(k+1)$.

We consider $S_\varepsilon := \{a_k^\varepsilon, 1 \leq k \leq 4\} \subset \Omega$, for $\varepsilon \in \mathbb{C}$, $\mathcal{I}_\varepsilon := \mathcal{I}(S_\varepsilon)$.

In general we should consider $A \subset \mathbb{C}$ such that $0 \in \bar{A} \setminus A$ and study limits along A ; quite often we will use some compactness to ensure convergence and pass to a subsequence included in A . For simplicity, we will just write \lim or \lim_ε instead of $\lim_{\varepsilon \rightarrow 0, \varepsilon \in A}$.

We will write several sufficient conditions about convergence of ideals and Green functions in terms of the asymptotic directions defined by pairs of poles:

$$v_{ij}^\varepsilon := [a_j^\varepsilon - a_i^\varepsilon] \in \mathbb{P}^1\mathbb{C},$$

where $[\cdot]$ denotes the class in $\mathbb{P}^1\mathbb{C}$ of an element of $\mathbb{C}^2 \setminus \{(0, 0)\}$. Since $\mathbb{P}^1\mathbb{C}$ is compact, by restricting to an appropriate subsequence we assume $v_{ij} = \lim_{\varepsilon \rightarrow 0} v_{ij}^\varepsilon \in \mathbb{P}^1\mathbb{C}$, for $1 \leq i < j \leq 4$. When such convergence does not occur as $\varepsilon \rightarrow 0$ in an unrestricted fashion, one may consider the (possible) limits obtained from “convergent” subsequences, and conclude about global convergence by examining whether the partial limits coincide or not.

Let

$$\mathcal{D}^\varepsilon = \mathcal{D}(S_\varepsilon) := \{v_{ij}^\varepsilon \in \mathbb{P}^1\mathbb{C}, 1 \leq i < j \leq 4\}, \quad \mathcal{D} := \{v_{ij} \in \mathbb{P}^1\mathbb{C}, 1 \leq i < j \leq 4\}.$$

Given a subset $\tilde{S}_\varepsilon \subset S_\varepsilon$, we can define $\tilde{\mathcal{D}}_\varepsilon$ and $\tilde{\mathcal{D}}$ in a similar manner.

2.2. The generic 4-pole case.

Theorem 2.1. *Let S_ε satisfy*

$$(2.1) \quad \forall \tilde{S}_\varepsilon \subset S_\varepsilon \text{ with } \#\tilde{S}_\varepsilon = 3, \text{ then } \#\tilde{\mathcal{D}} \geq 2.$$

and

$$(2.2) \quad \forall k \in \{1, 2, 3, 4\}, \quad \#\{v_{km} \in \mathbb{P}^1\mathbb{C} : m \in \{1, 2, 3, 4\} \setminus \{k\}\} \geq 2,$$

then there exists $\lim_\varepsilon \mathcal{I}_\varepsilon = \mathcal{I}$, with $\mathfrak{M}_0^3 \subset \mathcal{I} \subset \mathfrak{M}_0^2$ and $\ell(\mathcal{I}) = 4$; and $\lim_\varepsilon G_\varepsilon = G_J = 2 \max(\log |z_1|, \log |z_2|) + O(1)$ depends only on Ω and not on \mathcal{I} .

2.3. Some singular cases. We will see how things change when we give up the second condition in Theorem 2.1.

Theorem 2.2. *Suppose that S_ε verifies condition (2.1), and*

$$(2.3) \quad \exists i \in I := \{1, 2, 3, 4\} \text{ s.t. } \#\{v_{ij} \in \mathbb{CP}^1 : j \in I \setminus \{i\}\} = 1,$$

then, after a linear change of variables, $\lim_{\varepsilon \rightarrow 0} \mathcal{I}(S_\varepsilon) = \mathcal{I}_0 := \langle z_1 z_2, z_2^2, z_1^3 \rangle$,

and

$$\liminf_\varepsilon G_\varepsilon \geq G_{\mathcal{I}_0}(z) = \max \{ \log |z_1 z_2|, 2 \log |z_2|, 3 \log |z_1| \} + O(1),$$

but there is no equality.

If the situation becomes even more singular, we can have more diverse limits for the ideals.

Theorem 2.3. *Suppose there exist a 3 point subset $\tilde{S}_\varepsilon \subset S_\varepsilon$ such that $\#\tilde{\mathcal{D}} = 1$. Then*

- (1) *If $\#\mathcal{D} \geq 3$, then, after an appropriate linear change of variables, $\lim_{\varepsilon \rightarrow 0} \mathcal{I}(S_\varepsilon) = \mathcal{I}_0$.*
- (2) *If $\#\mathcal{D} = 2$, then, after passing to a subsequence and an appropriate linear change of variables, $\lim_{\varepsilon \rightarrow 0} \mathcal{I}(S_\varepsilon) = \mathcal{I}_0$ or $= \mathcal{J}_0 := \langle z_1 z_2, z_1^2 + k z_2^2, z_1^3 \rangle$, for some $k \in \mathbb{C} \setminus \{0\}$.*

We suspect that the Green functions do admit a limit, but we haven't been able to determine it.

2.4. Upper and lower limits of ideals. We now formalize the notion of convergence of ideals using upper and lower limits.

Definition 2.4. [7]

- (i) $\liminf_{A \ni \varepsilon \rightarrow 0} \mathcal{I}_\varepsilon$ is the ideal consisting of all $f \in \mathcal{O}(\Omega)$ such that $f_\varepsilon \rightarrow f$ locally uniformly on Ω , as $\varepsilon \rightarrow 0$, where $f_\varepsilon \in \mathcal{I}_\varepsilon$.
- (ii) $\limsup_{A \ni \varepsilon \rightarrow 0} \mathcal{I}_\varepsilon$ is the ideal of $\mathcal{O}(\Omega)$ generated by all functions f such that $f_j \rightarrow f$ locally uniformly, as $j \rightarrow \infty$, for some sequence $\varepsilon_j \rightarrow 0$ in A and $f_j \in \mathcal{I}_{\varepsilon_j}$.
- (iii) If the two limits are equal, we say that the family \mathcal{I}_ε converges and write $\lim_{A \ni \varepsilon \rightarrow 0} \mathcal{I}_\varepsilon$ for the common value of the upper and lower limits.

This last notion of convergence is equivalent to convergence in the topology of the Douady space [7, Section 3]. Clearly, $\liminf_\varepsilon \mathcal{I}_\varepsilon \subset \limsup_\varepsilon \mathcal{I}_\varepsilon$ and so $\ell(\liminf_\varepsilon \mathcal{I}_\varepsilon) \geq \ell(\limsup_\varepsilon \mathcal{I}_\varepsilon)$. It also follows from [7, Lemmas 2.1 and 2.2] that $\ell(\limsup_\varepsilon \mathcal{I}_\varepsilon) \leq \limsup \ell(\mathcal{I}_\varepsilon)$ and $\ell(\liminf_\varepsilon \mathcal{I}_\varepsilon) \leq \liminf \ell(\mathcal{I}_\varepsilon)$.

Theorem 2.5. *Let \mathcal{I}_ε be a family of ideals based on N distinct points, so that $\ell(\mathcal{I}_\varepsilon) = N$, for any ε .*

- (i) *Let $\mathcal{I} := \limsup_\varepsilon \mathcal{I}_\varepsilon$. If $\ell(\mathcal{I}) \geq N$ (or equivalently $= N$), then $\lim_\varepsilon \mathcal{I}_\varepsilon = \mathcal{I}$.*
- (ii) *Let $\mathcal{I} := \liminf_\varepsilon \mathcal{I}_\varepsilon$. If $\ell(\mathcal{I}) \leq N$ (or equivalently $= N$), then $\lim_\varepsilon \mathcal{I}_\varepsilon = \mathcal{I}$.*

3. PROOF OF THEOREM 2.5

We will proceed by reducing everything to upper and lower limits of subspaces of a single finite-dimensional vector space.

We use multiindex notation, in particular if $\alpha, \beta \in \mathbb{N}^n$, $\alpha \leq \beta$ means $\alpha_j \leq \beta_j$, $1 \leq j \leq n$ (and the analogous definition for “ $<$ ”).

Let π_j denote the projection to the j -th coordinate axis. Passing to a subsequence if needed, $N_j := \#\pi_j(\{a_1^\varepsilon, \dots, a_N^\varepsilon\})$ is independent of ε . Let $\mathcal{N} := (N_1, \dots, N_n)$ and

$$P_\varepsilon := \pi_1(\{a_1^\varepsilon, \dots, a_N^\varepsilon\}) \times \dots \times \pi_n(\{a_1^\varepsilon, \dots, a_N^\varepsilon\}) \text{ (cartesian product)}$$

As in [7, Section 2], we now define a simpler sequence of ideals contained in each $\mathcal{I}_\varepsilon = \mathcal{I}(\{a_1^\varepsilon, \dots, a_N^\varepsilon\})$. Let $\mathcal{J}_\varepsilon := \mathcal{I}(P_\varepsilon)$. It is easy to see that $d := \ell(\mathcal{J}_\varepsilon) = \#P_\varepsilon = \prod_{j=1}^n N_j \leq N^n$, and [7, Lemma 2.3] gives

$$\lim_{\varepsilon \rightarrow 0} \mathcal{J}_\varepsilon = \mathcal{J} := \langle z_1^{N_1}, \dots, z_n^{N_n} \rangle = \left\{ f \in \mathcal{O}(\Omega) : \frac{\partial^\alpha f}{\partial z^\alpha} : \alpha < \mathcal{N} \right\}.$$

Claim. $\mathcal{O}/\mathcal{J}_\varepsilon \cong \mathbb{C}^d \cong \mathcal{O}/\mathcal{J}$.

Indeed, denote by $b_j^{i,\varepsilon}$ the elements of $\pi_j(\{a_1^\varepsilon, \dots, a_N^\varepsilon\})$. For $\alpha \leq \mathcal{N}$, set $\Psi_\alpha(z) = z^\alpha$, and for $\zeta \in \mathbb{C}$, $1 \leq j \leq n$, $0 \leq k \leq N_j - 1$,

$$\varphi_{k,j}^\varepsilon(\zeta) := \prod_{i=1}^k (\zeta - b_j^{i,\varepsilon})$$

Let

$$\Psi_\alpha^\varepsilon(z) := \prod_{j=1}^n \varphi_{\alpha_j,j}^\varepsilon(z_j).$$

Since all the $b_j^{i,\varepsilon}$ tend to 0, it is easy to see that for ε small enough (including $\varepsilon = 0$) the system $\{\Psi_\alpha^\varepsilon, \alpha < \mathcal{N}\}$ is linearly independent.

Let $[\cdot]_\varepsilon$ (resp. $[\cdot]$) denote the class of a function in $\mathcal{O}/\mathcal{J}_\varepsilon$ (resp. \mathcal{O}/\mathcal{J}). The natural projection from $\text{Span}\{\Psi_\alpha^\varepsilon, \alpha \leq \mathcal{N}\}$ to $\mathcal{O}/\mathcal{J}_\varepsilon$ is injective, thus bijective, and $\{[\Psi_\alpha^\varepsilon]_\varepsilon, \alpha < \mathcal{N}\}$ is a basis of $\mathcal{O}/\mathcal{J}_\varepsilon$. Then the linear map defined by $\Phi_\varepsilon\left([\Psi_\alpha^\varepsilon]_\varepsilon\right) = [\Psi_\alpha]$, for $\alpha < \mathcal{N}$, is the required isomorphism.

Lemma 3.1. *Suppose that $\lim_\varepsilon f_\varepsilon = f$, uniformly on compacta of Ω . Then, in the finite dimensional vector space \mathcal{O}/\mathcal{J} , $\{\Phi_\varepsilon([f_\varepsilon]_{\mathcal{J}_\varepsilon})\} \rightarrow [f]$ as $\varepsilon \rightarrow 0$.*

Proof. There is a unique choice of coefficients $c_\alpha^\varepsilon(f)$ such that $f_\varepsilon = \sum_{\alpha < \mathcal{N}} c_\alpha^\varepsilon(f_\varepsilon) \Psi_\alpha^\varepsilon + h_\varepsilon$, with $h_\varepsilon \in \mathcal{J}_\varepsilon$. It will be enough to show that $c_\alpha^\varepsilon(f_\varepsilon) \rightarrow c_\alpha(f)$ as $\varepsilon \rightarrow 0$, for each α .

By rescaling, we might assume that $\overline{\mathbb{D}}^n \subset \Omega$. One can prove by induction on n (or deduce as an easy special case from the beginning of [11]) that if $|\varepsilon|$ is small enough, then

$$c_\alpha^\varepsilon(f_\varepsilon) = \frac{1}{(2i\pi)^n} \int_{(\partial\mathbb{D})^n} \frac{f_\varepsilon(z_1, \dots, z_n)}{\Psi_\alpha^\varepsilon(z)} \frac{dz_1}{z_1} \cdots \frac{dz_n}{z_n},$$

and one sees that those integrals converge towards the required limit. \square

We define upper and lower limits for families of subspaces in a finite dimensional vector space \mathbb{C}^d by first choosing a norm on it. Since they are equivalent, we may as well choose a euclidean norm, and we do.

Then let L_ε be a family of subspaces of \mathbb{C}^d such that $\dim L_\varepsilon = k$, for any ε . Let $K_\varepsilon := L_\varepsilon \cap \overline{B}(0; 1)$. We can define the upper and lower limits of L_ε by $\liminf_\varepsilon L_\varepsilon := \text{Span}(\liminf K_\varepsilon)$ where $\liminf K_\varepsilon$ is taken in the sense of the Hausdorff distance between compacta (and inclusion as an order relation), and analogously $\limsup_\varepsilon L_\varepsilon := \text{Span}(\limsup K_\varepsilon)$.

Proposition 3.2. (1) $\limsup_{\varepsilon \rightarrow 0} \Phi_\varepsilon(\mathcal{I}_\varepsilon/\mathcal{J}_\varepsilon) = (\limsup \mathcal{I}_\varepsilon)/\mathcal{J}$,
 (2) $\liminf_{\varepsilon \rightarrow 0} \Phi_\varepsilon(\mathcal{I}_\varepsilon/\mathcal{J}_\varepsilon) = (\liminf \mathcal{I}_\varepsilon)/\mathcal{J}$.

Proof. To prove that $\limsup \mathcal{I}_\varepsilon/\mathcal{J} \subset \limsup (\Phi_\varepsilon(\mathcal{I}_\varepsilon/\mathcal{J}_\varepsilon))$, it is enough to consider elements $[f]$ where f is in a generating system of $\limsup \mathcal{I}_\varepsilon$. So there exist $(\varepsilon_j)_{j \in \mathbb{Z}_+}$, $\varepsilon_j \rightarrow 0$ as $j \rightarrow +\infty$ and $f_j \in \mathcal{I}_{\varepsilon_j}$ such that $f_j \rightarrow f$ uniformly on compacta of Ω . Proposition 3.1 implies that $\Phi_{\varepsilon_j}([f_j]_{\mathcal{J}_{\varepsilon_j}}) \rightarrow [f]$.

Conversely, take $g \in \mathcal{O}/\mathcal{J}$ such that there exists $(\varepsilon_j)_{j \in \mathbb{Z}_+}$, $\varepsilon_j \rightarrow 0$ as $j \rightarrow +\infty$ and $g_j \in \mathcal{I}_{\varepsilon_j}$ such that $\|\Phi_{\varepsilon_j}([g_j]_{\mathcal{J}_{\varepsilon_j}}) - [g]\| \rightarrow 0$ as $j \rightarrow +\infty$. Then $|C_\alpha^{\varepsilon_j}(g_j) - C_\alpha(g)| \rightarrow 0$ for any $\alpha < N$. We can write

$$g(z) = \sum_{\alpha \in \Gamma} C_\alpha(g) z^\alpha + \sum_{j=1}^n z_j^{N_j} R_j(z) \text{ and}$$

$$[g_j(z)]_{\mathcal{J}_{\varepsilon_j}} = \sum_{\alpha \in \Gamma} C_\alpha^{\varepsilon_j}(g_j) [\Psi_\alpha^{\varepsilon_j}(z)]_{\mathcal{J}_{\varepsilon_j}} \in \mathcal{I}_{\varepsilon_j}/\mathcal{J}_{\varepsilon_j}.$$

Set

$$f_j(z) := \sum_{\alpha \in \Gamma} C_\alpha^{\varepsilon_j}(g_j) \Psi_\alpha^{\varepsilon_j}(z) + \sum_{j=1}^n \prod_{i=1}^{N_j} (z_j - b_j^{i, \varepsilon_j}) R_j(z).$$

Then $f_j \in \mathcal{I}_{\varepsilon_j}$ and $f_j \rightarrow g$ uniformly on compacta of Ω .

Since the g 's as above form a generating system for $\limsup (\Phi_\varepsilon(\mathcal{I}_\varepsilon/\mathcal{J}_\varepsilon))$, we are done.

The proof for \liminf is analogous and we omit it. \square

The proof of our theorem then reduces to an elementary fact about families of finite dimensional spaces.

Lemma 3.3. *Let (L_ε) be a family of vector subspaces of \mathbb{C}^d such that $\dim L_\varepsilon = k \leq n$, for any ε .*

- (1) *If $\dim(\limsup_{\varepsilon \rightarrow 0} L_\varepsilon) = k$, then $\liminf_{\varepsilon \rightarrow 0} L_\varepsilon = \limsup_{\varepsilon \rightarrow 0} L_\varepsilon$.*
- (2) *If $\dim(\liminf_{\varepsilon \rightarrow 0} L_\varepsilon) = k$, then $\liminf_{\varepsilon \rightarrow 0} L_\varepsilon = \limsup_{\varepsilon \rightarrow 0} L_\varepsilon$.*

Proof. (1). Let L stand for $\limsup L_\varepsilon$. For any $\eta \in (0, \frac{1}{2})$, there exists $\varepsilon_\eta > 0$ such that $|\varepsilon| \leq \varepsilon_\eta$ implies that $L_\varepsilon \cap \overline{B}(0; 1)$ is contained in an η -neighborhood of $L \cap \overline{B}(0; 1)$. So the orthogonal projection of $L_\varepsilon \cap \overline{B}(0; 1)$ to L must contain at least the ball $L \cap \overline{B}(0; (1 - \eta^2)^{1/2})$, and any point of $L \cap \overline{B}(0; 1)$ is a distance at most $\eta + 1 - (1 - \eta^2)^{1/2}$ from $L_\varepsilon \cap \overline{B}(0; 1)$, so $L \subset \liminf_\varepsilon L_\varepsilon$.

(2). Let $L := \liminf L_\varepsilon$. If we had $\limsup L_\varepsilon \not\subset L$, then $\limsup L_\varepsilon \supsetneq L$ and we can pick a unit vector $v \in \limsup L_\varepsilon \cap L^\perp$. We can find a sequence $\varepsilon_j \rightarrow 0$ and vectors $v_j \rightarrow v$, $v_j \in L_{\varepsilon_j}$. L_{ε_j} must also contain k vectors $e_1^{\varepsilon_j}, \dots, e_k^{\varepsilon_j}$ close to the vectors in an orthonormal basis e_1, \dots, e_k of L . For j large enough, the system $e_1^{\varepsilon_j}, \dots, e_k^{\varepsilon_j}, v_j$ will have to be linearly independent, which contradicts $\dim L_{\varepsilon_j} = k$. \square

4. PROOFS OF THEOREMS 2.1, 2.2 AND 2.3

4.1. Previous results.

Definition 4.1. *A (point based) ideal is a complete intersection ideal if and only if it admits a set of n generators, where n is the dimension of the ambient space.*

The main result of [7], Theorem 1.11, states:

Theorem 4.2. *Let $\mathcal{I}_\varepsilon = \mathcal{I}(S_\varepsilon)$, where S_ε is a set of N points all tending to 0 and assume that $\lim_{\varepsilon \rightarrow 0} \mathcal{I}_\varepsilon = \mathcal{I}$. Then $(G_{\mathcal{I}_\varepsilon})$ converges to $G_{\mathcal{I}}$ locally uniformly on $\Omega \setminus \{0\}$ if and only if \mathcal{I} is a complete intersection ideal.*

The following was also defined in [7].

Definition 4.3. *The family of ideals $(\mathcal{I}_\varepsilon)$ satisfies the Uniform Complete Intersection Condition if for any ε , there exists a map Ψ_0 and maps Ψ_ε from a neighborhood of $\overline{\Omega}$ to \mathbb{C}^n such that Ψ_0 is proper from Ω to $\Psi_0(\Omega)$, and*

- (1) $\{a_j^\varepsilon, 1 \leq j \leq N\} = \Psi_\varepsilon^{-1}\{0\}$, for all ε ;
- (2) For all $\varepsilon \neq 0$, $1 \leq j \leq N$ and z in a neighborhood of a_j^ε ,

$$|\log \|\Psi_\varepsilon(z)\| - \log \|z - a_j^\varepsilon\|| \leq C(\varepsilon) < \infty;$$

(3) $\lim_{\varepsilon \rightarrow 0} \Psi_\varepsilon = \Psi = (\Psi^1, \dots, \Psi^n)$, uniformly on $\overline{\Omega}$.

Notice that the first two conditions imply $\mathcal{I}_\varepsilon = \langle \Psi_\varepsilon^1, \dots, \Psi_\varepsilon^n \rangle$.

This is [7, Theorem 1.8]:

Theorem 4.4. *Let $(\mathcal{I}_\varepsilon)$ be a family of ideals satisfying the uniform complete intersection condition, set $S_\varepsilon = V(\mathcal{I}_\varepsilon)$ and $\mathcal{I} = \langle \Psi^1, \dots, \Psi^n \rangle$. Then*

- (1) $\lim_{\varepsilon \rightarrow 0} \mathcal{I}_\varepsilon = \mathcal{I}$,
- (2) $\lim_{\varepsilon \rightarrow 0} G_\varepsilon = G_{\mathcal{I}}$, and the convergence is locally uniform on $\Omega \setminus \{0\}$.

4.2. Proof of Theorem 2.1.

Let $l_{ij}^\varepsilon, 1 \leq i < j \leq 4$ be the (normalized) equations of the lines through $a_i^\varepsilon, a_j^\varepsilon$ and $l_{ij} := \lim_{\varepsilon \rightarrow 0} l_{ij}^\varepsilon, 1 \leq i < j \leq 4$. Set

$$\mathcal{L}^\varepsilon := \{f_1^\varepsilon := l_{12}^\varepsilon \cdot l_{34}^\varepsilon; f_2^\varepsilon := l_{13}^\varepsilon \cdot l_{24}^\varepsilon; f_3^\varepsilon := l_{14}^\varepsilon \cdot l_{23}^\varepsilon\} \subset \mathcal{I}(S_\varepsilon),$$

and $f_j := \lim_{\varepsilon \rightarrow 0} f_j^\varepsilon, j = 1, 2, 3$.

We will prove that under the hypotheses of the theorem, there exists $i \neq j \in \{1, 2, 3\}$ such that if $\Psi_0 := (f_i, f_j)$, then $\Psi_0^{-1}(0) = \{0\}$. (One can see that the hypotheses are necessary for this to happen [3, Remarque 4.1.2, p. 66]). Then we conclude using Theorem 4.4 with $\Psi_\varepsilon := f_i^\varepsilon f_j^\varepsilon$. Notice that since Ψ_0 is homogeneous of degree 2 and $\|\Psi_0\|$ is bounded and bounded away from 0 on the unit sphere, then $\log \|\Psi_0\| = \log \|z\|^2 + O(1)$, and the same estimate holds for $G_{\mathcal{I}}$. An application of the generalized maximum principle of Rashkovskii and Sigurdsson [8, Lemma 4.1] shows that the limit does not depend on the particular value of $\|\Psi_0\|$: there is only one maximal plurisubharmonic function with boundary values 0 on $\partial\Omega$ and a singularity equivalent to $\log \|z\|^2$.

We proceed with the proof that we can find an “independent” pair of f_i ’s.

Case 1: For any three point subset $\tilde{S}_\varepsilon \subset S_\varepsilon$, the set of limit directions satisfies $\#\tilde{\mathcal{D}} = 3$. So whenever $\{i, j\}$ and $\{i', j'\}$ have an element in common, l_{ij} is independent from $l_{i'j'}$ and so for any $1 \leq k < k' \leq 3$, f_k and $f_{k'}$ have no common factor. So $\Psi_0^{-1}(0) = \{0\}$.

Case 2: Suppose that there exists a three point subset $S'_\varepsilon \subset S_\varepsilon$ such that the set \mathcal{D}' of limit directions satisfies $\#\mathcal{D}' = 2$. Without loss of generality, $S'_\varepsilon = \{a_1^\varepsilon, a_2^\varepsilon, a_3^\varepsilon\} \subset S_\varepsilon$.

Write v_{ij} for the direction in \mathbb{P}^1 defined by l_{ij} . With our hypothesis, we may assume $v_{23} = v_{12} \neq v_{13}$. It will be convenient to write

$$\begin{aligned} A_1 &:= \{v_{13}, v_{24}\} \cap \{v_{12}, v_{34}\}, \\ A_2 &:= \{v_{13}, v_{24}\} \cap \{v_{14}, v_{23}\}, \\ A_3 &:= \{v_{12}, v_{34}\} \cap \{v_{14}, v_{23}\}. \end{aligned}$$

So here $A_3 \neq \emptyset$. We will show that there exists $p \in \{1, 2\}$ such that $A_p = \emptyset$ (and thus the corresponding couple of function f_i will be without a common factor, and the proof concluded).

Suppose $A_1 \neq \emptyset$. Since $v_{23} = v_{12} \neq v_{13}$, by (2.2), $v_{12} \neq v_{24}$. Consequently, $v_{34} \in \{v_{13}, v_{24}\}$.

We study A_2 . Since $v_{23} = v_{12} \neq v_{24}$, $v_{23} \notin \{v_{13}, v_{24}\}$. So we need to study v_{14} .

Case 2.1: $v_{34} = v_{13}$.

Then (2.1) implies that $v_{14} \neq v_{13} = v_{34}$. We will see that $v_{14} = v_{24}$ is impossible. For this, we need to take some coordinates.

Using translations, we may assume $a_1^\varepsilon = 0 \in \mathbb{D}^2$, for any ε . Choose vectors $\tilde{v}_{ij} \in \mathbb{C}^2$ such that $|\tilde{v}_{ij}| = 1$ and $[\tilde{v}_{ij}] = v_{ij} \in \mathbb{P}^1\mathbb{C}$, $1 \leq i < j \leq 4$. Since $v_{23} = v_{12} \neq v_{13}$, we can choose an invertible linear map Φ such that $[\Phi(\tilde{v}_{12})] = [1 : 0]$, $[\Phi(\tilde{v}_{13})] = [0 : 1]$. So we can study $\Phi(S_\varepsilon)$, where

$$\begin{aligned} \Phi(a_1^\varepsilon) &= b_1^\varepsilon = (0, 0), \quad \Phi(a_2^\varepsilon) = b_2^\varepsilon = (\rho_2(\varepsilon), \eta_2(\varepsilon)), \\ \Phi(a_3^\varepsilon) &= b_3^\varepsilon = (\eta_3(\varepsilon), \rho_3(\varepsilon)), \quad \Phi(a_4^\varepsilon) = b_4^\varepsilon = (\alpha(\varepsilon), \beta(\varepsilon)) \end{aligned}$$

in which all coordinates tend to 0 and $\lim_{\varepsilon \rightarrow 0} \eta_j(\varepsilon)/\rho_j(\varepsilon) = 0$, $j = 2, 3$. We retain the notation $v_{ij} \in \mathbb{P}^1\mathbb{C}$, $1 \leq i < j \leq 4$, and $v_{ij} := \lim_{\varepsilon} v_{ij}^\varepsilon$ where this last is the direction of the line through b_i^ε and b_j^ε . Let

$$\gamma(\varepsilon) := \frac{\rho_3(\varepsilon) - \eta_2(\varepsilon)}{\eta_3(\varepsilon) - \rho_2(\varepsilon)},$$

then $v_{23}^\varepsilon = [1 : \gamma(\varepsilon)]$. Since $v_{23} = v_{12} = [1 : 0]$, $\lim_{\varepsilon \rightarrow 0} \gamma(\varepsilon) = 0$. Thus

$$\lim_{\varepsilon \rightarrow 0} \frac{\rho_3(\varepsilon)}{\rho_2(\varepsilon)} = \lim_{\varepsilon \rightarrow 0} \frac{\gamma(\varepsilon) - \frac{\eta_2(\varepsilon)}{\rho_2(\varepsilon)}}{\gamma(\varepsilon) \cdot \frac{\eta_3(\varepsilon)}{\rho_3(\varepsilon)} - 1} = 0.$$

Assume now that $v_{14} = v_{24}$. Then $[1 : 0] = v_{12} \neq v_{14} = v_{24} \neq v_{34} = [0 : 1]$. Write $v_{14} = [1 : \ell]$, i.e. $\beta/\alpha \rightarrow \ell \neq 0, \infty$. Consider ρ_2/α . If $\|\rho_2/\alpha\| \leq C_2 < \infty$, as $\varepsilon \rightarrow 0$ (or even along a subsequence $\varepsilon_k \rightarrow 0$), then

$$\frac{\alpha - \eta_3}{\beta - \rho_3} = \frac{1 - \frac{\eta_3}{\rho_3} \cdot \frac{\rho_3}{\rho_2} \cdot \frac{\rho_2}{\alpha}}{\frac{\beta}{\alpha} - \frac{\rho_3}{\rho_2} \cdot \frac{\rho_2}{\alpha}} \rightarrow \ell \neq 0, \quad \text{as } \varepsilon \rightarrow 0.$$

This contradicts $\lim_{\varepsilon \rightarrow 0} [\alpha - \eta_3 : \beta - \rho_3] = v_{34} = v_{13} = [0 : 1]$. Therefore we have $\alpha/\rho_2 \rightarrow 0$, so

$$\frac{\beta - \eta_2}{\alpha - \rho_2} = \frac{\frac{\beta}{\alpha} \cdot \frac{\alpha}{\rho_2} - \frac{\eta_2}{\rho_2}}{\frac{\alpha}{\rho_2} - 1} \rightarrow 0, \text{ as } \varepsilon \rightarrow 0.$$

This contradicts $\lim_{\varepsilon \rightarrow 0} [\alpha - \rho_2 : \beta - \eta_2] = v_{24} = v_{14} \neq v_{12} = [1 : 0]$. This is the contradiction we sought.

Case 2.2: $v_{34} = v_{24}$.

In an analogous way, we will see that $A_2 = \emptyset$. We still have $v_{23} \notin \{v_{13}, v_{24}\}$. By condition (2.2), on a $v_{14} \neq v_{24} = v_{34}$. We still use the coordinates above.

Suppose that $v_{14} = v_{13} = [0 : 1]$. This implies $\alpha/\beta \rightarrow 0$. If $0 < \|\rho_2/\beta\| \leq C_4 < \infty$ as $\varepsilon \rightarrow 0$,

$$\frac{\alpha - \eta_3}{\beta - \rho_3} = \frac{\alpha/\beta - \eta_3/\rho_3 \cdot \rho_3/\rho_2 \cdot \rho_2/\beta}{1 - \rho_3/\rho_2 \cdot \rho_2/\beta} \rightarrow 0, \text{ as } \varepsilon \rightarrow 0.$$

This contradicts $v_{34} = v_{24} \neq v_{23} = [0 : 1]$. Thus $\beta/\rho_2 \rightarrow 0$, therefore

$$\frac{\beta - \eta_2}{\alpha - \rho_2} = \frac{\beta/\rho_2 - \eta_2/\rho_2}{\alpha/\beta \cdot \beta/\rho_2 - 1} \rightarrow 0, \text{ as } \varepsilon \rightarrow 0.$$

This contradicts $v_{24} = v_{34} \neq v_{23} = [1 : 0]$. So $v_{14} \neq v_{13}$.

In a similar way, we can prove that if $A_2 \neq \emptyset$, then $A_1 = \emptyset$. \square

To finish the proof of Theorem 2.1, we need to prove the statements about the limit ideal. General properties of convergence show that $\ell(\mathcal{I}) = 4$ and the form of the generators show that $\mathcal{I} \subset \mathfrak{M}_0^2$. It remains to prove that $\mathcal{I} \supset \mathfrak{M}_0^3$, which is a consequence of a more general fact.

Proposition 4.5. *Suppose that all the directions in $\mathcal{D}(S_\varepsilon)$ admit a limit, and that $\#\mathcal{D} \geq 2$. Then $\mathfrak{M}_0^3 \subset \liminf_{\varepsilon \rightarrow 0} \mathcal{I}_\varepsilon$. Furthermore,*

$$\limsup_{\varepsilon} \mathcal{I}_\varepsilon \subset \mathfrak{M}_0^2.$$

Proof. \mathfrak{M}_0^3 is invariant under invertible linear maps. Since $\#\mathcal{D} \geq 2$, there exists $i \in \{1, 2, 3, 4\}$ such that $v_{ik} \neq v_{ik'}$, with $k \neq k'$ and $k, k' \in \{1, 2, 3, 4\} \setminus \{i\}$; otherwise it is easy to show that all directions are equal, in contradiction with the hypothesis.

Without loss of generality, assume $v_{12} \neq v_{13}$ and after a linear transformation, $v_{12} = [1 : 0]$, $v_{13} = [0 : 1]$.

We reduce ourselves by translations to the case $a_1^\varepsilon = (0, 0)$. Let $a_2^\varepsilon = (\rho_2(\varepsilon), \delta_2(\varepsilon))$ and $a_3^\varepsilon - a_1^\varepsilon = (\delta_3(\varepsilon), \rho_3(\varepsilon))$, where $\delta_j(\varepsilon) = o(\rho_j(\varepsilon))$,

$j = 2, 3$. Let $a_4^\varepsilon = (x_4(\varepsilon), y_4(\varepsilon))$ tending to $(0,)$. For any ε , set

$$\psi_1^\varepsilon := \left[z_1 - x_1(\varepsilon) - \frac{\delta_3(\varepsilon)}{\rho_3(\varepsilon)}(z_2 - x_2(\varepsilon)) \right] \left[z_1 - x_1(\varepsilon) - \rho_2(\varepsilon) \right] \left[z_1 - x_1(\varepsilon) - x_4(\varepsilon) \right],$$

$$\psi_2^\varepsilon := \left[z_1 - x_1(\varepsilon) - \frac{\delta_3(\varepsilon)}{\rho_3(\varepsilon)}(z_2 - x_2(\varepsilon)) \right] \left[z_1 - x_1(\varepsilon) - \rho_2(\varepsilon) \right] \left[z_2 - x_2(\varepsilon) - y_4(\varepsilon) \right],$$

$$\psi_3^\varepsilon := \left[z_1 - x_1(\varepsilon) - \frac{\delta_3(\varepsilon)}{\rho_3(\varepsilon)}(z_2 - x_2(\varepsilon)) \right] \left[z_2 - x_2(\varepsilon) - \frac{\delta_2(\varepsilon)}{\rho_2(\varepsilon)}(z_1 - x_1(\varepsilon)) \right] \left[z_2 - x_2(\varepsilon) - y_4(\varepsilon) \right]$$

$$\psi_4^\varepsilon := \left[z_2 - x_2(\varepsilon) - \frac{\delta_2(\varepsilon)}{\rho_2(\varepsilon)}(z_1 - x_1(\varepsilon)) \right] \left[z_2 - x_2(\varepsilon) - \rho_3(\varepsilon) \right] \left[z_2 - x_2(\varepsilon) - y_4(\varepsilon) \right].$$

Then $\psi_j^\varepsilon \in \mathcal{I}_\varepsilon$, $1 \leq j \leq 4$, and, with uniform convergence on compacta of Ω ,

$$z_1^3 = \lim_{\varepsilon \rightarrow 0} \psi_1^\varepsilon \in \liminf_{\varepsilon \rightarrow 0} \mathcal{I}_\varepsilon$$

$$z_1^2 z_2 = \lim_{\varepsilon \rightarrow 0} \psi_2^\varepsilon \in \liminf_{\varepsilon \rightarrow 0} \mathcal{I}_\varepsilon$$

$$z_1 z_2^2 = \lim_{\varepsilon \rightarrow 0} \psi_3^\varepsilon \in \liminf_{\varepsilon \rightarrow 0} \mathcal{I}_\varepsilon$$

$$z_2^3 = \lim_{\varepsilon \rightarrow 0} \psi_4^\varepsilon \in \liminf_{\varepsilon \rightarrow 0} \mathcal{I}_\varepsilon.$$

Thus $\mathfrak{M}_0^3 = \langle z_1^3, z_1^2 z_2, z_1 z_2^2, z_2^3 \rangle \subset \liminf_{\varepsilon \rightarrow 0} \mathcal{I}_\varepsilon$.

To get the other inclusion, we make the same normalizations (using the fact that \mathfrak{M}_0^2 is invariant under invertible linear transformations, too). Write $\tilde{S}_\varepsilon = \{a_1^\varepsilon, a_2^\varepsilon, a_3^\varepsilon\}$. By [7, Theorem 1.12, i], $\lim_{\varepsilon \rightarrow 0} \mathcal{I}(\tilde{S}_\varepsilon) = \mathfrak{M}_0^2$. Since $\mathcal{I}_\varepsilon \subset \mathcal{I}(\tilde{S}_\varepsilon)$, $\limsup_{\varepsilon \rightarrow 0} \mathcal{I}_\varepsilon \subset \lim_{\varepsilon \rightarrow 0} \mathcal{I}(\tilde{S}_\varepsilon) = \mathfrak{M}_0^2$. \square

4.3. Proof of Theorem 2.2. The fact that the limit inferior of the Green functions is greater than the Green function of the ideal, but not equal to it, follows from Theorem 4.2 since here \mathcal{I}_0 has 3 generators.

Remark. It would be desirable to have a better estimate of the limits of Green functions. Some explicit computations were carried out in [3, Section 4.3], using the methods from [9]. It concerned the family of poles given by $S_\varepsilon := \{(0; 0), (\varepsilon; 0), (0; \varepsilon), (\gamma\varepsilon; 0)\}$, with $\gamma \neq 1$. Since the family is homogeneous in ε , in particular is given by a hyperplane section of a (singular) holomorphic curve, [9, Example 5.8] shows that the limit of the Green functions does exist.

The following estimates are obtained:

- (1) $\lim_{\varepsilon \rightarrow 0} G_{\mathcal{I}_\varepsilon}(z) \geq 2 \log \|z\| + O(1)$, for $z_2 \neq 0$;
- (2) $\lim_{\varepsilon \rightarrow 0} G_{\mathcal{I}_\varepsilon}(z) \geq \frac{5}{3} \log \|z\| + O(1)$, for $z_1 z_2^2 (z_1 + z_2)(z_1 + \gamma z_2) \neq 0$.

This is far from a complete answer, even in this case, but the computations involved are getting increasingly tedious.

We now proceed with the proof of convergence of the family of ideals.

As before, we may assume $a_1^\varepsilon = 0 \in \Omega$. Since $\#\tilde{\mathcal{D}} \geq 2$, for any three-point set $\tilde{S}_\varepsilon \subset S_\varepsilon$, $\#\mathcal{D} \geq 2$. Without loss of generality, assume $v_{12} \neq v_{13}$. By (2.3), we may assume that for $i = 2$, $v_{12} = v_{23} = v_{24}$.

Then we claim that $\#\mathcal{D} \geq 3$. Indeed, if we had $\#\mathcal{D} = 2$, then $\mathcal{D} = \{v_{12}, v_{13}\}$. Three cases may occur.

-) If $v_{14} = v_{12}$, then $v_{12} = v_{14} = v_{24}$. This contradicts (2.1).
-) If $v_{34} = v_{12}$, then $v_{23} = v_{34} = v_{24}$. This contradicts (2.1).
-) Si $v_{14} = v_{34} = v_{13}$, then this contradicts (2.1).

This proves the claim.

We can choose an invertible linear map $\Phi : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ such that

$$[\Phi(\tilde{v}_{12})] = [1 : 0] \text{ and } [\Phi(\tilde{v}_{13})] = [0 : 1],$$

where $\tilde{v}_{12}, \tilde{v}_{13} \in \mathbb{C}^2$ are chosen so that $\|\tilde{v}_{12}\| = \|\tilde{v}_{13}\| = 1$ and $[\tilde{v}_{12}] = v_{12}$, $[\tilde{v}_{13}] = v_{13}$. Then

$$\Phi(S_\varepsilon) = S'_\varepsilon = \{b_1^\varepsilon = (0, 0), b_2^\varepsilon, b_3^\varepsilon, b_4^\varepsilon\}.$$

For this new system $v_{12} = [1 : 0] \neq v_{13} = [0 : 1]$. We can choose $l_{ij}^\varepsilon(z)$, normalized equations of the lines through the pairs of points b_i^ε and b_j^ε , $1 \leq i < j \leq 4$ such that $\lim_{\varepsilon \rightarrow 0} l_{12}^\varepsilon(z) = \lim_{\varepsilon \rightarrow 0} l_{23}^\varepsilon(z) = \lim_{\varepsilon \rightarrow 0} l_{24}^\varepsilon(z) = z_2$ and $\lim_{\varepsilon \rightarrow 0} l_{13}^\varepsilon(z) = z_1$. This implies

$$z_1 z_2 = \lim_{\varepsilon \rightarrow 0} l_{13}^\varepsilon(z) l_{24}^\varepsilon(z) \in \liminf_{\varepsilon} \mathcal{I}_\varepsilon,$$

$$z_1^3 = \lim_{\varepsilon \rightarrow 0} l_{13}^\varepsilon(z) [z_1 - z_1(b_2^\varepsilon)] [z_1 - z_1(b_4^\varepsilon)] \in \liminf_{\varepsilon} \mathcal{I}_\varepsilon.$$

So $\langle z_1 z_2, z_1^3 \rangle \subset \mathcal{I}_*$.

Since $\#\mathcal{D} \geq 3$, there exists $(i, j) \in \{(1, 3), (1, 4), (3, 4)\}$ such that $v_{ij}^\varepsilon \rightarrow [1 : t]$, with $t \neq 0, \infty$. So $\lim_{\varepsilon \rightarrow 0} l_{ij}^\varepsilon(z) = z_2 - t z_1 \in \liminf_{\varepsilon} \mathcal{I}_\varepsilon$. This implies

$$z_2^2 = \lim_{\varepsilon \rightarrow 0} (l_{ij}^\varepsilon(z) l_{km}^\varepsilon(z) + t l_{13}^\varepsilon(z) l_{24}^\varepsilon(z)) \in \liminf_{\varepsilon} \mathcal{I}_\varepsilon,$$

since $2 \in \{k, m\} = \{1, 2, 3, 4\} \setminus \{i, j\}$. Thus $\mathcal{I}_0 := \langle z_1 z_2, z_2^2, z_1^3 \rangle \subset \liminf_{\varepsilon} \mathcal{I}$, with $\ell(\mathcal{I}_0) = 4$. By Theorem 2.5, $\lim_{\varepsilon \rightarrow 0} \mathcal{I}(S_\varepsilon) = \mathcal{I}_0$. \square

4.4. Proof of Theorem 2.3. By the hypothesis $\#\mathcal{D} \geq 2$, we may assume $v_{12} \neq v_{13}$. Just as in the proof of Theorem 2.2, we perform a translation to reduce ourselves to $a_1^\varepsilon = (0, 0)$, and we choose a linear map Φ so that we are reduced to $v_{12} = [1 : 0] \neq v_{13} = [0 : 1]$. We adopt the same notation $S'_\varepsilon = \{b_k^\varepsilon, 1 \leq k \leq 4\}$.

Since there is a 3 point subset $\tilde{S}'_\varepsilon \subset S'_\varepsilon$ such that $\#\tilde{\mathcal{D}}' = 1$, we may assume that $\tilde{S}'_\varepsilon = \{1, 2, 4\}$, so $v_{12} = v_{14} = v_{24} = [1 : 0]$. Again we may

choose line equations so that $\lim_{\varepsilon \rightarrow 0} l_{12}^\varepsilon(z) = \lim_{\varepsilon \rightarrow 0} l_{14}^\varepsilon(z) = \lim_{\varepsilon \rightarrow 0} l_{24}^\varepsilon(z) = z_2$ and $\lim_{\varepsilon \rightarrow 0} l_{13}^\varepsilon(z) = z_1$.

The proof in case (i) can then be completed exactly as the proof of Theorem 2.2 above.

Case (ii): $\#\mathcal{D} = 2$.

Either there exists $(i, j) \in \{(2, 3), (3, 4)\}$ such that $v_{ij}^\varepsilon \rightarrow v_{ij} = [1 : 0]$ or $v_{23} = v_{34} = [0 : 1]$.

Case (ii.1): there exists $(i, j) \in \{(2, 3), (3, 4)\}$ such that $v_{ij}^\varepsilon \rightarrow v_{ij} = [1 : 0]$.

Then $\lim_{\varepsilon \rightarrow 0} l_{ij}^\varepsilon(z) = z_2$. Then, again,

$$z_2^2 = \lim_{\varepsilon \rightarrow 0} l_{ij}^\varepsilon(z) l_{km}^\varepsilon(z) \in \liminf_{\varepsilon} \mathcal{I}_\varepsilon,$$

since $2 \in \{k, m\} = \{1, 2, 3, 4\} \setminus \{i, j\}$. Again, as in the proof of Theorem 2.2, we find that $\lim_{\varepsilon \rightarrow 0} \mathcal{I}(S_\varepsilon) = \mathcal{I}_0$.

Case (ii.2): $v_{23} = v_{34} = [0 : 1]$.

Thus $v_{13} = v_{23} = v_{34} = [0 : 1]$ and $v_{12} = v_{14} = v_{24} = [1 : 0]$.

As in [2], where systems of three points tending to the origin along a single direction are considered, we reparametrize $\{b_1^\varepsilon, b_2^\varepsilon, b_4^\varepsilon\}$ in such a way that $|\varepsilon| = \|b_2^\varepsilon - b_1^\varepsilon\|$ and choose a coordinate system depending on ε such that

$$b_1^\varepsilon = (0, 0), b_2^\varepsilon = (\varepsilon, 0), b_4^\varepsilon = (\rho(\varepsilon), \delta(\varepsilon)\rho(\varepsilon)) \text{ where } 0 < |\rho(\varepsilon)| \leq \frac{1}{2}|\varepsilon|, \delta(\varepsilon) \rightarrow 0,$$

as $\varepsilon \rightarrow 0$. Denote $b_3^\varepsilon = (\alpha(\varepsilon), \beta(\varepsilon))$. Since $v_{13}^\varepsilon = [\alpha(\varepsilon) : \beta(\varepsilon)] \rightarrow [0 : 1]$, $\lim_{\varepsilon \rightarrow 0} \frac{\alpha(\varepsilon)}{\beta(\varepsilon)} = 0$. We will write $\rho = \rho(\varepsilon), \delta = \delta(\varepsilon), \alpha = \alpha(\varepsilon), \beta = \beta(\varepsilon)$. For δ small enough, set

$$\tilde{\delta} := \frac{\delta}{1 - \frac{\alpha}{\beta}\delta}, \quad \tilde{\rho} := \rho(1 - \frac{\alpha}{\beta}\delta).$$

Clearly $\tilde{\delta}, \tilde{\rho} \rightarrow 0$. Furthermore,

$$\frac{\tilde{\delta}}{\tilde{\rho} - \varepsilon} = \frac{\delta/(\rho - \varepsilon)}{\left(1 - \frac{\alpha}{\beta} \frac{\delta\rho}{\rho - \varepsilon}\right)(1 - \frac{\alpha}{\beta}\delta)},$$

so if $\lim_{\varepsilon \rightarrow 0} \frac{\delta}{\rho - \varepsilon} = m$, then $\lim_{\varepsilon \rightarrow 0} \frac{\tilde{\delta}}{\tilde{\rho} - \varepsilon} = m$. Consider the following biholomorphism (a small perturbation of the identity map):

$$\Phi_{1,\varepsilon} : \mathbb{C}^2 \longrightarrow \mathbb{C}^2, \quad z \mapsto \Phi_{1,\varepsilon}(z) = \left(z_1 - \frac{\alpha}{\beta}z_2, z_2\right),$$

Then $\Phi_{1,\varepsilon}(S'_\varepsilon) = S_{1,\varepsilon} = \{(0,0), (\varepsilon,0), (\tilde{\rho}, \tilde{\delta}\tilde{\rho}), (0,\beta)\}$. Since $v_{23} = [0:1]$ and $v_{13} = [0:1]$, $|\alpha - \varepsilon| \ll \frac{1}{2}|\beta|$ et $|\alpha| \ll \frac{1}{2}|\beta|$. So $|\varepsilon| \leq |\alpha - \varepsilon| + |\alpha| \ll |\beta|$.

The proof is concluded with the following result. Notice that this limit ideal in case (ii.2) is deduced from \mathcal{I}_0 by exchanging the coordinates z_1 and z_2 , so is again equivalent to it by a linear invertible map.

Proposition 4.6. *Let $S_\varepsilon = \{(0,0), (\varepsilon,0), (\rho, \delta\rho), (0,\beta)\}$ tend to $(0,0)$ as $\varepsilon \rightarrow 0$, with $\rho := \rho(\varepsilon), \delta := \delta(\varepsilon), \beta := \beta(\varepsilon)$ and $0 < |\rho| \leq \frac{1}{2}|\varepsilon|$, $|\varepsilon| \ll |\beta|$. Then*

$$(i) \quad \text{If } \lim_{\varepsilon \rightarrow 0} \frac{\delta}{\rho - \varepsilon} = m \neq \infty, \lim_{\varepsilon \rightarrow 0} \mathcal{I}(S_\varepsilon) = \mathcal{I}_0 := \langle z_1 z_2, z_2^2, z_1^3 \rangle.$$

$$ii) \quad \text{If } \lim_{\varepsilon \rightarrow 0} \frac{\delta}{\varepsilon} = \infty, \text{ we have two cases:}$$

$$ii.1) \quad \text{If } \lim_{\varepsilon \rightarrow 0} \frac{\rho - \varepsilon}{\delta\beta} = k \notin \{0, \infty\}, \text{ then } \lim_{\varepsilon \rightarrow 0} \mathcal{I}(S_\varepsilon) = \mathcal{J}_0 := \langle z_1 z_2, z_1^2 + k z_2^2, z_1^3 \rangle.$$

$$ii.2) \quad \text{If } \lim_{\varepsilon \rightarrow 0} \frac{\rho - \varepsilon}{\delta\beta} = k \in \{0, \infty\}, \text{ then } \lim_{\varepsilon} \mathcal{I}(S_\varepsilon) = \mathcal{I}_0, \text{ if } k = \infty \text{ and } \mathcal{I}_1 := \langle z_1 z_2, z_1^2, z_2^3 \rangle, \text{ if } k = 0.$$

Proof. Since $|\varepsilon| \ll |\beta|$,

$$z_1 z_2 = \lim_{\varepsilon \rightarrow 0} (z_2 - \rho z_1) \left[z_1 + \frac{\varepsilon}{\beta} z_2 - \varepsilon \right] \in \liminf_{\varepsilon} \mathcal{I}_\varepsilon, \text{ and}$$

$$z_1^3 = \lim_{\varepsilon \rightarrow 0} z_1 (z_1 - \rho)(z_1 - \varepsilon) \in \liminf_{\varepsilon} \mathcal{I}_\varepsilon.$$

Thus $\langle z_1 z_2, z_1^3 \rangle \subset \liminf_{\varepsilon} \mathcal{I}_\varepsilon$.

Now we need to look at various cases separately.

i) Since $\lim_{\varepsilon \rightarrow 0} \frac{\delta}{\rho - \varepsilon} = m \neq \infty$ and the polynomial

$$Q_\varepsilon(z) := \frac{\delta\varepsilon}{\rho - \varepsilon}(\delta\rho - \beta)z_1 - \beta z_2 - \frac{\delta}{\rho - \varepsilon}(\delta\rho - \beta)z_1^2 + z_2^2 \in \mathcal{I}(S_\varepsilon),$$

we obtain $z_2^2 = \lim_{\varepsilon \rightarrow 0} Q_\varepsilon(z) \in \liminf_{\varepsilon} \mathcal{I}_\varepsilon$. So $\mathcal{I}_0 := \langle z_1 z_2, z_2^2, z_1^3 \rangle \subset \liminf_{\varepsilon} \mathcal{I}_\varepsilon$. Since $\ell(\mathcal{I}_0) = 4$, applying Theorem 2.5 $\lim_{\varepsilon \rightarrow 0} \mathcal{I}(S_\varepsilon) = \mathcal{I}_0$.

ii) Since $0 < |\rho| \leq \frac{1}{2}|\varepsilon|$, $\frac{|\varepsilon|}{2} \leq |\rho - \varepsilon| \leq |\varepsilon|$, so if $\lim_{\varepsilon \rightarrow 0} \frac{\delta}{\varepsilon} = \infty$,

$$\lim_{\varepsilon \rightarrow 0} \frac{\rho - \varepsilon}{\delta} = \lim_{\varepsilon \rightarrow 0} \frac{\rho - \varepsilon}{\varepsilon} \frac{\varepsilon}{\delta} = 0.$$

We consider two subcases:

ii.1) Suppose $\lim_{\varepsilon \rightarrow 0} \frac{\rho - \varepsilon}{\delta\beta} = k \notin \{0, \infty\}$. Consider the polynomial

$$(4.1) \quad P_\varepsilon(z) := -\varepsilon z_1 + \frac{\rho - \varepsilon}{\delta\beta} \frac{\beta}{\frac{\delta\rho}{\beta} - 1} z_2 + z_1^2 - \frac{\rho - \varepsilon}{\delta\beta} \frac{1}{\frac{\delta\rho}{\beta} - 1} z_2^2.$$

We can check that $P_\varepsilon(z) \in \mathcal{I}(S_\varepsilon)$. Since $|\delta\rho| \ll |\rho| \leq \frac{1}{2}|\varepsilon| \ll |\beta|$, il vient $\frac{\delta\rho}{\beta} \rightarrow 0$. So

$$z_1^2 + kz_2^2 = \lim_{\varepsilon \rightarrow 0} P_\varepsilon(z) \in \liminf_\varepsilon \mathcal{I}_\varepsilon.$$

Thus $\mathcal{J}_0 := \langle z_1 z_2, z_1^2 + kz_2^2, z_1^3 \rangle \subset \liminf_\varepsilon \mathcal{I}_\varepsilon$. But the class $[z_1^2] = [z_1^2 + kz_2^2] - k[z_2^2] = -k[z_2^2] \in \mathcal{O}(\Omega)/\mathcal{J}_0$, thus $\mathcal{O}(\Omega)/\mathcal{J}_0 = \text{Span}\{[1], [z_1], [z_2], [z_2^2]\}$ and $\ell(\mathcal{J}_0) = 4$. Using Theorem 2.5, we conclude $\lim_{\varepsilon \rightarrow 0} \mathcal{I}(S_\varepsilon) = \mathcal{J}_0$.

ii.2) Suppose $\lim_{\varepsilon \rightarrow 0} \frac{\rho - \varepsilon}{\delta\beta} = k \in \{0, \infty\}$. Analogously to (4.1), consider the polynomial

$$R_\varepsilon(z) := \frac{\delta\beta}{\varepsilon} \left(\frac{\delta\rho}{\beta} - 1 \right) \frac{\varepsilon}{\rho - \varepsilon} \varepsilon z_1 - \beta z_2 - \frac{\delta\beta}{\varepsilon} \left(\frac{\delta\rho}{\beta} - 1 \right) \frac{\varepsilon}{\rho - \varepsilon} z_1^2 + z_2^2.$$

We can check that $R_\varepsilon(z) \in \mathcal{I}(S_\varepsilon)$. If $k = \infty$, then $|\delta\beta| \ll |\rho - \varepsilon| \ll |\delta|$, and

$$\lim_{\varepsilon \rightarrow 0} \frac{\delta\beta}{\varepsilon} = \lim_{\varepsilon \rightarrow 0} \frac{\delta\beta}{\rho - \varepsilon} \frac{\rho - \varepsilon}{\varepsilon} = 0.$$

Thus $z_2^2 = \lim_{\varepsilon \rightarrow 0} P_\varepsilon(z) \in \liminf_\varepsilon \mathcal{I}_\varepsilon$. Then $\mathcal{I}_0 \subset \liminf_\varepsilon \mathcal{I}_\varepsilon$. Since $\ell(\mathcal{I}_0) = 4$, using Theorem 2.5, we conclude $\lim_{\varepsilon \rightarrow 0} \mathcal{I}(S_\varepsilon) = \mathcal{I}_0$.

Finally, if $k = 0$, $|\rho - \varepsilon| \ll |\delta\beta| \ll |\delta|$. From (4.1) we deduce $z_1^2 = \lim_{\varepsilon \rightarrow 0} P_\varepsilon(z) \in \liminf_\varepsilon \mathcal{I}_\varepsilon$. In addition, $z_2^3 = \lim_{\varepsilon \rightarrow 0} z_2(z_2 - \delta\rho)(z_2 - \beta) \in \liminf_\varepsilon \mathcal{I}_\varepsilon$.

Therefore $\mathcal{I}_1 := \langle z_1 z_2, z_1^2, z_2^3 \rangle \subset \liminf_\varepsilon \mathcal{I}_\varepsilon$. We conclude as before. \square

REFERENCES

- [1] J.-P. Demailly, *Mesures de Monge-Ampère et mesures pluriharmoniques*, Math. Z. 194 (1987), 519–564.
- [2] Duong Quang Hai, P. J. Thomas, *Limit Of Three-Point Green Functions : The Degenerate Case*, Serdica 40 (2014), 99–110.
- [3] Duong Quang Hai, *Limites d'idéaux de fonctions holomorphes et de fonctions de Green pluricomplexes*, Ph. D. thesis, Université Toulouse III Paul Sabatier, July 8th, 2013, 94 pp.
- [4] L. Hörmander, *An Introduction to Complex Analysis in Several Variables*, Third Edition (revised), Mathematical Library, Vol. 7, North Holland, Amsterdam-New York-Oxford-Tokyo, 1990.

- [5] P. Lelong, *Fonction de Green pluricomplexe et lemmes de Schwarz dans les espaces de Banach*, J. Math. Pures Appl. 68 (1989), 319–347.
- [6] L. Lempert, *La métrique de Kobayashi et la représentation des domaines sur la boule*, Bull. Soc. Math. France 109 (1981), no. 4, 427–474.
- [7] J. I. Magnusson, A. Rashkovskii, R. Sigurdsson, P. J. Thomas, *Limits of multipole pluricomplex Green functions*, Int. J. Math. 23 (2012), no. 6, DOI: 10.1142/S0129167X12500656.
- [8] A. Rashkovskii, R. Sigurdsson, *Green functions with singularities along complex spaces*, Int. J. Math. 16 (2005), no. 4, 333–355.
- [9] A. Rashkovskii, P. J. Thomas, *Powers of ideals and convergence of Green functions with colliding poles*, Int. Math. Res. Not. IMRN (2014) 1253–1272.
- [10] P. J. Thomas, *Green vs. Lempert functions: a minimal example*, Pacific J. Math., Vol. 257 (2012), no. 1, pp. 189–197.
- [11] A. K. Tsikh, *Multidimensional Residues and Their Applications*, Translations of Mathematical Monographs, Vol. 103, American Mathematical Society, Providence, 1992.
- [12] V.P. Zahariuta, *Spaces of analytic functions and maximal plurisubharmonic functions*. D. Sci. Dissertation, Rostov-on-Don, 1984.

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